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Spectral analysis of nonselfadjoint Schrödinger operators with a matrix potential

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Abstract

Dissipative Schrödinger operators with a matrix potential are studied in $L_2((0,\infty); E)$ ($\dim E = n < \infty$) which are extension of a minimal symmetric operator L_0 with defect index (n, n). A selfadjoint dilation of a dissipative operator is constructed, using the Lax–Phillips scattering theory, the spectral analysis of a dilation is carried out, and the scattering matrix of a dilation is founded. A functional model of the dissipative operator is constructed and its characteristic function's analytic properties are determined, theorems on the completeness of eigenvectors and associated vectors of a dissipative Schrödinger operator are proved.
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1. Introduction

It is known [1] that the spectral theory of dissipative operators is based on the theory of dilations with application of operator models. The characteristic function is placed in this theory; it carries the complete information on the spectral properties of the operator. According to the scheme of Lax and Phillips, the computation of the characteristic function

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in terms of the Weyl–Titcmarsh function of the corresponding selfadjoint operators is preceded by the construction and investigation of a selfadjoint dilation and the corresponding scattering problem matrix [2]. This method has already been used in many investigation [3–5].

In this paper, we apply this approach to the study of maximal dissipative Schrödinger operators in the space $L_2((0,\infty);E)$ that extensions of a symmetric operators with defect index (n,n).

2. Selfadjoint dilation of the dissipative Schrödinger operator

Let E be n-dimensional $(n<\infty)$ Euclidean space. We denote by $L_2((0,\infty);E)$ the Hilbert space of vector-valued functions with values in E.

We consider the system of differential equations of Schrödinger type

$$\ell(y) = -y'' + \frac{\nu^2 - 1/4}{x^2} y(x) + Q(x)y(x) = \lambda y(x), \quad 0 < x < \infty, \tag{1}$$

where $0 \leqslant \nu < 1$ and $Q(x) = Q^*(x)$, and elements of the matrix $Q(x)$ is a Lebesgue measurable and locally integrable functions on $(0,\infty)$. We denote by L_0 the closure of the minimal operator generated by the expression $\ell(y)$, and by D_0 its domain. Further, we denote by D the set of all functions $y(x)$ from $L_2((0,\infty);E)$ whose first derivatives are locally absolutely continuous in $(0,\infty)$ and $\ell(y) \in L_2((0,\infty);E)$. Then D is the domain of the maximal operator L generated by the expression $\ell(y)$, and $L = L_0^*$ [6]. The symmetric operator L_0 has defect index (n,n). It is known that the defect number $\operatorname{def} L_0$ of the operator L_0 is determined by the formula $\operatorname{def} L_0 = \operatorname{def} L_0^- + \operatorname{def} L_0^+ - 2n$ [6]. Since $0 \leqslant \nu < 1$, $\operatorname{def} L_0^- = 2n$ and then $\operatorname{def} L_0^+ = n$.

Let $v_1(x), v_2(x) \in D$ be the operator (matrix) solutions of the equation $\ell(y) = 0$ satisfying the initial conditions

$$v_1(1) = I, \quad v_1'(1) = 0; \qquad v_2(1) = 0, \quad v_2'(1) = I.$$

Clearly, $v_1(x)$ and $v_2(x)$ are linearly independent and their Wronskians are equal to

$$W[v_1, v_2]_x = v_2^{*\prime}(x)v_1(x) - v_2^*(x)v_1'(x) = W[v_1, v_2]_1 = I.$$

We adopt the following notation:

$$(Wy)(x) = \begin{pmatrix} (W_1 y)(x) \\ (W_2 y)(x) \end{pmatrix} = \begin{pmatrix} v_2^{*\prime}(x)y(x) - v_2^*(x)y'(x) \\ -v_1^{*\prime}(x)y(x) + v_1^*(x)y'(x) \end{pmatrix}.$$

We denote by Γ_1 and Γ_2 the linear mappings of D into E defined by

$$\Gamma_1 y = -(W_1 y)(0), \quad \Gamma_2 y = (W_2 y)(0), \quad y \in D. \tag{2}$$

Then we have the following assertion.

Lemma 1. *The triple (E, Γ_1, Γ_2) is space of boundary values of the operator L_0 for the linear mappings defined by* (2).

From [7], following theorem is obtained.

Theorem 2. *For any contraction K in E the restriction of the operator L to the set of vector-valued functions $y \in D$ which satisfy the boundary condition*

$$(K - I)\Gamma_1 y + i(K + I)\Gamma_2 y = 0 \tag{3}$$

is a maximal dissipative extension of the operator L_0. Conversely, every maximally dissipative extension of the operator L_0 is restriction of L to the set of vector-valued function $y \in D$ satisfying (3)*, and the contraction K is uniquely determined by the extension. The condition* (3) *defines selfadjoint extension if K is unitary.*

We study the maximal dissipative operators L_K generated by the expression $\ell(y)$ and the boundary condition (3), where K is a strict contraction (i.e., $\|K\|_E < 1$) on E. Since K is a strict contraction the boundary condition (3) is equivalent to

$$\Gamma_2 y + B\Gamma_1 y = 0, \tag{4}$$

where $B = -i(K+I)^{-1}(K-I)$, $\operatorname{Im} B > 0$ and $-K$ is the Cayley transform of the dissipative operator B. Then, we denote by $\tilde{L}_B$ $(= L_K)$ the operator generated by the expression $\ell(y)$ and the boundary condition (4).

In order to construct a selfadjoint dilation of the dissipative operator $\tilde{L}_B$ $(= L_K)$, we add to $H := L_2((0,\infty); E)$ the "incoming" and the "outgoing" channels $D_\pm := D_+ = L_2((0,\infty); E)$ and $D_- = L_2((-\infty,0); E)$. We form the orthogonal sum $\mathcal{H} = D_- \oplus H \oplus D_+$ and call it the basic Hilbert space of the dilation. We then extend the construction to a selfadjoint operator $\mathcal{L}_B$ on a space $\mathcal{H}$ as follows.

Let $\varphi^- \in W_2^1((-\infty,0); E)$, $\varphi^+ \in W_2^1((0,\infty); E)$, $u \in D$, where W_2^1 is Sobolev space. We define the operator of $\mathcal{L}_B$ on the set of elements $D(\mathcal{L}_B)$ in $\mathcal{H}$ satisfying the boundary conditions

$$\begin{aligned} &\Gamma_2 u + B\Gamma_1 u = C\varphi^-(0), \\ &\Gamma_2 u + B^*\Gamma_1 u = C\varphi^+(0), \end{aligned} \tag{5}$$

where $C^2 = 2\operatorname{Im} B$, $C > 0$, is generated by

$$\mathcal{L}\langle \varphi^-, u, \varphi^+\rangle = \left\langle i\frac{d\varphi^-}{d\xi}, \ell(u), i\frac{d\varphi^+}{d\xi}\right\rangle. \tag{6}$$

We then have

Theorem 3. *The operator $\mathcal{L}_B$ is selfadjoint in $\mathcal{H}$ and is a selfadjoint dilation of the dissipative operator $\tilde{L}_B$ $(= L_K)$.*

Proof. We first prove that $\mathcal{L}_B$ is symmetric in $\mathcal{H}$. Let $U_1 = \langle \varphi_1^-, u_1, \varphi_1^+\rangle \in D(\mathcal{L}_B)$ and $U_2 = \langle \varphi_2^-, u_2, \varphi_2^+\rangle \in D(\mathcal{L}_B)$. Since $\operatorname{def} L_0 = (n,n)$, for every $u(x), v(x) \in D$ if we consider that

$$[u,v]_\infty = \lim_{x\to\infty}\bigl[\bigl(u(x), v'(x)\bigr)_E - \bigl(v'(x), u(x)\bigr)_E\bigr] = 0,$$

we obtain

$$(\mathcal{L}_B U_1, U_2)_{\mathcal{H}} - (U_1, \mathcal{L}_B U_2)_{\mathcal{H}} = 0.$$

Thus, $\mathcal{L}_B$ is a symmetric operator. Therefore, to prove that $\mathcal{L}_B$ is selfadjoint, it suffices for us to show that $\mathcal{L}_B^* \subset \mathcal{L}_B$.

Take $U_2 \in D(\mathcal{L}_B^*)$. Let $\mathcal{L}_B^* U_2 = U_2^* = \langle \varphi_2^{*-}, u_2^*, \varphi_2^{*+} \rangle \in \mathcal{H}$, so that

$$(\mathcal{L}_B U_1, U_2)_{\mathcal{H}} = \big(U_1, \mathcal{L}_B^* U_2\big) = \big(U_1, U_2^*\big)_{\mathcal{H}}. \tag{7}$$

By choosing elements with suitable components as the $U_1 \in D(\mathcal{L}_B)$ in (7), it is not difficult to show that $\varphi_2^- \in W_2^1((-\infty, 0); E)$, $\varphi_2^+ \in W_2^1((0, \infty); E)$, $u_2 \in D$ and $U_2^* = \mathcal{L} U_2$ the operation $\mathcal{L}$ is defined by (6). Consequently, (7) is obtained from the equality of $(\mathcal{L} U_1, U_2)_{\mathcal{H}} = (U_1, \mathcal{L} U_2)_{\mathcal{H}}$ for all $U_1 \in D(\mathcal{L}_B^*)$. Furthermore, $U_2 \in D(\mathcal{L}_B^*)$ satisfies the conditions

$$\Gamma_2 u_2 + B\Gamma_1 u_2 = C\varphi_2^-(0),$$
$$\Gamma_2 u_2 + B^*\Gamma_1 u_2 = C\varphi_2^+(0).$$

Hence, $\mathcal{L}_B^* \subseteq \mathcal{L}_B$ or $\mathcal{L}_B^* = \mathcal{L}_B$, i.e., the operator $\mathcal{L}_B$ is selfadjoint in $\mathcal{H}$.

The selfadjoint operator $\mathcal{L}_B$ generates on $\mathcal{H}$ a unitary group $\mathcal{U}_t = \exp(i\mathcal{L}_B t)$ $(0 < t < \infty)$. We show by $P : \mathcal{H} \to H$ and $P_1 : H \to \mathcal{H}$ the mappings agreeing with the expressions $P : \langle \varphi^-, u, \varphi^+ \rangle \to u$ and $P_1 : u \to \langle 0, u, 0 \rangle$, respectively. Let $Z_t = P\mathcal{U}_t P_1$, $t \geqslant 0$, by using unitary group $\mathcal{U}_t$. The family $\{Z_t\}$ of operators which has the generator A is a strongly continuous semigroup of completely nonunitary contractions on $\mathcal{H}$ which is denoted by

$$Ay = \lim_{t \to +0} (it)^{-1}(Z_t y - y).$$

The operator A is dissipative, that is $\operatorname{Im} A \geqslant 0$ and operator $\mathcal{L}_B$ is named the selfadjoint dilation of A. So, we should denote $A = \tilde{L}_B$ and then it is shown that $\mathcal{L}_B$ is a selfadjoint dilation of $\tilde{L}_B$. To do this, it is sufficient to control that

$$P(\mathcal{L}_B - \lambda I)^{-1} P_1 y = (\tilde{L}_B - \lambda I)^{-1} y, \quad y \in H,\ \operatorname{Im}\lambda < 0. \tag{8}$$

To prove (8), we set the expression $(\mathcal{L}_B - \lambda I)^{-1} P_1 y = g = \langle \psi^-, z, \psi^+ \rangle$. Hence it is shown that $(\mathcal{L}_B - \lambda I)g = P_1 y$ and $(\mathcal{L}_B - \lambda I)\langle \psi^-, z, \psi^+ \rangle = \langle 0, y, 0 \rangle$. And then we obtain as follows:

$$\left(i\frac{d\psi^-}{d\xi}, \ell(z), i\frac{d\psi^+}{d\xi} \right) - \lambda(\psi^-, z, \psi^+) = \langle 0, y, 0 \rangle,$$
$$\psi^-(\xi) = \psi^-(0)e^{-i\lambda\xi}, \qquad \psi^+(\xi) = \psi^+(0)e^{-i\lambda\xi}$$

and

$$z = (\tilde{L}_B - \lambda I)^{-1} y + z_\lambda.$$

Since $g \in D(\mathcal{L}_B)$ and hence $\psi^- \in L_2((-\infty, 0); E)$ it follows that $\psi^-(0) = 0$. Here z_λ is a solution of the homogeneous equation

$$-y''(x) + \frac{\nu^2 - 1/4}{x^2} y(x) + Q(x)y(x) - \lambda y(x) = 0$$

satisfying the boundary of $\Gamma_2 z + B\Gamma_1 z = 0$. Therefore, $z \in D(\mathcal{L}_B)$ and since a point λ with $\operatorname{Im}\lambda < 0$ cannot be an eigenvalue of a dissipative operator, it follows that $z_\lambda \equiv 0$ and then $z = (\tilde{L}_B - \lambda I)^{-1} y$. Applying the mapping P, we get (8).

We easily show that $A = \tilde{L}_B$. Indeed, by (8),

$$(\tilde{L}_B - \lambda I)^{-1} = P(\mathcal{L}_B - \lambda I)^{-1} P_1 = -iP \int_0^\infty \mathcal{U}_t e^{-i\lambda t}\, dt\, P_1$$
$$= -i \int_0^\infty Z_t e^{-i\lambda t}\, dt = (A - \lambda I)^{-1}$$

is denoted and it is clear that $\tilde{L}_B = A$. Theorem 3 is proved. □

3. Scattering theory of dilations and functional model of dissipative operators

We show the matrix-valued solutions of the equation $\ell(y) = \lambda y$ $(0 < x < \infty)$ satisfying following conditions by $\varphi_1(x, \lambda)$ and $\varphi_2(x, \lambda)$:

$$(W_1\varphi_1)(0) = 0, \qquad (W_2\varphi_1)(0) = -I,$$
$$(W_1\varphi_2)(0) = I, \qquad (W_2\varphi_2)(0) = 0.$$

The Weyl–Titchmarsh matrix-valued function $M_\infty(\lambda)$ of the selfadjoint operator L_∞ generated by the boundary condition $(W_1 y)(0) = 0$ is uniquely determined from the condition

$$\int_0^\infty \left[\varphi_2^*(x, \lambda) + M_\infty^*(\lambda)\varphi_1^*(x, \lambda)\right]\left[\varphi_2(x, \lambda) + \varphi_1(x, \lambda)M_\infty(\lambda)\right] dx < \infty.$$

So, $M_\infty(\lambda)$ is not generally a meromorphic function on $\mathbb{C}$, but it is a holomorphic function in case of $\operatorname{Im}\lambda \neq 0$ and $\operatorname{Im}\lambda \cdot \operatorname{Im} M_\infty(\lambda) \leqslant 0$. We suppose that matrix-valued function $M_\infty(\lambda)$ is meromorphic on $\mathbb{C}$. This condition is equivalent that any selfadjoint extensions of the operator L_0 has a purely discrete spectrum.

We consider the matrix-valued function

$$S_B(\lambda) = (\operatorname{Im} B)^{-1/2}\big(M_\infty(\lambda) + B\big)\big(M_\infty(\lambda) + B^*\big)^{-1}(\operatorname{Im} B)^{1/2}. \tag{9}$$

$S_B(\lambda)$ is meromorphic function on $\mathbb{C}$ and its all poles are in the lower half-plane. $\|S_B(\lambda)\|_E \leqslant 1$ for all $\operatorname{Im}\lambda > 0$ and $S_B(\lambda)$ is an unitary matrix for all $\lambda \in \mathbb{R}$.

It is important property of the unitary group that the Lax–Phillips scheme is applicable to it [2]. Namely, group $\{\mathcal{U}_t\}$ has the incoming $D_- = \langle L_2((-\infty, 0); E), 0, 0\rangle$ and outgoing $D_+ = \langle 0, 0, L_2((0, \infty); E)\rangle$ subspaces satisfying the following properties:

(1) $\mathcal{U}_t D_- \subset D_-$, $t \leqslant 0$; $\mathcal{U}_t D_+ \subset D_+$, $t \geqslant 0$;

(2) $\bigcap_{t \leqslant 0} \mathcal{U}_t D_- = \bigcap_{t \geqslant 0} \mathcal{U}_t D_+ = \{0\}$;

(3) $\overline{\bigcup_{t \leqslant 0} \mathcal{U}_t D_-} = \overline{\bigcup_{t \geqslant 0} \mathcal{U}_t D_+} = \mathcal{H}$;

(4) $D_- \perp D_+$.

These properties may be proved. Property (4) is obvious. In order to prove (1) and (2) properties, we set $R_\lambda = (\mathcal{L}_B - \lambda I)^{-1}$ for subspace D_+ (the proof for D_- is familiar with D_+). For all λ in the half-plane $\operatorname{Im}\lambda < 0$ and for all $f = \langle 0, 0, \varphi\rangle \in D_+$ we have that

$$R_\lambda f = \left\langle 0, 0, -ie^{-i\lambda\xi} \int_0^\xi e^{i\lambda s}\varphi(s)\,ds \right\rangle.$$

Hence, it is shown that $R_\lambda f \in D_+$ for all $f = \langle 0, 0, \varphi\rangle \in D_+$. Therefore, if $g \perp D_+$, then

$$\begin{aligned} 0 = (R_\lambda f, g)_{\mathcal{H}} &= \big((\mathcal{L}_B - \lambda I)^{-1} f, g\big)_{\mathcal{H}} = \left(-i\int e^{i(\mathcal{L}_B-\lambda I)t} f\,dt, g\right) \\ &= \left(-i\int e^{i\mathcal{L}_B t}e^{-i\lambda t} f\,dt, g\right) = -i\int_0^\infty e^{-i\lambda t}(\mathcal{U}_t f, g)_{\mathcal{H}}\,dt, \quad \operatorname{Im}\lambda < 0, \end{aligned}$$

which implies that $(\mathcal{U}_t f, g) = 0$ for all $t \geqslant 0$. Consequently, $\mathcal{U}_t f \in D_+$ for $t \geqslant 0$, and property (1) is approved.

In order to prove (2), we define the mappings $\mathcal{P}$ and $\mathcal{P}_1$ as follows:

$$\begin{aligned} &\mathcal{P}: \mathcal{H} \to H \quad \text{and} \quad \mathcal{P}_1: H \to D_+, \\ &\langle \varphi^-, y, \varphi^+\rangle \to \varphi^+, \qquad \varphi \to \langle 0, 0, \varphi\rangle. \end{aligned}$$

We take into consider that the semigroup of isometrics $U_t^+ = \mathcal{P}\mathcal{U}_t\mathcal{P}_1$ $(t \geqslant 0)$ is the one-sided shift on H. In fact, we know that the generator of the one-sided shift semigroup V_t on H defined as follows:

$$V_t\varphi(\xi) = \varphi(\xi - t), \quad \xi > t, \quad \text{and} \quad V_t\varphi(\xi) = 0 \quad \text{for } 0 \leqslant \xi \leqslant t,$$

is the differentiation operator $i(d/d\xi)$ which has boundary condition $\varphi(0) = 0$. On the other hand, the generator A of the semigroup of isometrics U_t^+, $t \geqslant 0$, is the operator

$$A\varphi = \mathcal{P}\mathcal{L}_B\mathcal{P}_1\varphi = \mathcal{P}\mathcal{L}_B\langle 0, 0, \varphi\rangle = \mathcal{P}\left\langle 0, 0, i\frac{d\varphi}{d\xi}\right\rangle = i\frac{d\varphi}{d\xi},$$

where $\varphi \in W_2^1((0,\infty); E)$ and $\varphi(0) = 0$. But since a semigroup is uniquely determined by its generator, $U_t^+ = V_t$ and then

$$\bigcap_{t\geqslant 0} \mathcal{U}_t D_+ = \left\langle 0, 0, \bigcap_{t\geqslant 0} V_t\big(L_2(0,\infty); E\big)\right\rangle = \{0\}.$$

To prove property (3), let us prove the following lemma.

Lemma 4. *The operator $\tilde{L}_B$ is totally nonselfadjoint* (*simple*).

Proof. Let $K \subset H$ be a nontrivial subspace on which $\tilde{L}_B$ induces a selfadjoint operator $\tilde{L}'_B$ (i.e., the subspace K in invariant with respect to the semigroup of isometrics $V_t = \exp(i\tilde{L}_B)$, $V_t^* = \exp(-i\tilde{L}_B^*)$, $t > 0$, $V_t^{*-1} = V_t$).

Our goal is show that the subspace K is equal to $\{0\}$. If $f \in K \cap D(\tilde{L}_B) = D(\tilde{L}'_B)$, then $f \in D(\tilde{L}^*_B)$ and

$$0 = \frac{d}{dt}\big\|\exp(i\tilde{L}_B t)f\big\|_H^2 = -2\big(\operatorname{Im} BW_1\big(\exp(i\tilde{L}_B t)f\big)(0), W_1\big(\exp(i\tilde{L}_B t)\big)f(0)\big)_H.$$

Since $f \in D(\tilde{L}'_B)$, operator $\tilde{L}'_B$ satisfies the above condition. From this, for the eigenvectors $v_\lambda(x)$ of $\tilde{L}_B$ lying in K that are eigenvectors of $\tilde{L}'_B$, we obtain $(W_1 v_\lambda)(0) = 0$. Consider the boundary condition $\Gamma_2 y + B\Gamma_1 y = 0$, $(W_2 y)(0) - B(W_1 y)(0) = 0$, $\operatorname{Im} B > 0$, we get $(W_2 v_\lambda)(0) = 0$ and $v_\lambda(x) \equiv 0$. As Weyl–Titchmarsh matrix-valued function $M_\infty(\lambda)$ is meromorphic on $\mathbb{C}$, the resolvent $\mathcal{R}_\lambda(L_\infty)$ of selfadjoint operator L_∞ and the resolvent $\mathcal{R}_\lambda(\tilde{L}'_B)$ of the selfadjoint operator $\tilde{L}'_B$ are compact operators. Therefore, the spectrum of $\tilde{L}'_B$ is purely discrete. Then, it is shown that $K = \{0\}$ with theorem on expansion in eigenvectors of the selfadjoint operator $\tilde{L}'_B$, that is $\tilde{L}_B$ is simple. The lemma is proved. □

Now we establish

$$H_- = \overline{\bigcup_{t \geqslant 0} \mathcal{U}_t D_-} \quad \text{and} \quad H_+ = \overline{\bigcup_{t \leqslant 0} \mathcal{U}_t D_+}.$$

Lemma 5. $\mathcal{H} = H_- + H_+$.

Proof. We consider property (1) of subspaces $D_\pm$. The subspace $\mathcal{H}' = \mathcal{H} \ominus (H_- + H_+)$ is invariant relative to the unitary group $\{\mathcal{U}_t\}$ where H' is subspace of H, and $\mathcal{H}'$ is expressed as $\mathcal{H}' = \langle 0, H', 0\rangle$. Therefore, if the subspace $\mathcal{H}'$ (and H') were nontrivial, then the unitary group $\{\mathcal{U}'_t\}$ restricted of $\tilde{L}_B$ to H' would be a selfadjoint operator in H'. But, since operator $\tilde{L}'_B$ is simple (Lemma 4), $H' = \{0\}$, i.e., $\mathcal{H}' = \{0\}$. The lemma is proved. □

In the scheme of Lax–Phillips scattering theory, the scattering matrix is defined by using the theory of spectral representation. We proceed to their construction and on this way we also prove property (3) for subspaces D_- and D_+.

We denote Weyl solution of the equation $\ell(y) = \lambda y$ with

$$\theta(x,\lambda) = \varphi_2(x,\lambda) + \varphi_1(x,\lambda)M_\infty(\lambda).$$

Let

$$U^-_{\lambda j}(x,\xi,\zeta) = \big\langle e^{-i\lambda\xi}e_j, \theta^*(x,\lambda)\big(M^*_\infty(\lambda) + B\big)^{-1}Ce_j, S^*_B e^{-i\lambda\varsigma}e_j\big\rangle, \quad j = 1,2,\ldots,n,$$

where $e_1, e_2, \ldots, e_n$ is an orthonormal basis for E. $U^-_{\lambda j}$ $(j = 1,2,\ldots,n)$ satisfy the equation $\mathcal{L}U = \lambda U$ and the boundary conditions (5).

With the help of the vectors $U^-_{\lambda j}$ we define the transformation $\mathcal{F}_- : f \to \tilde{f}_-(\lambda)$ by

$$(\mathcal{F}_- f)(\lambda) := \tilde{f}_-(\lambda) := \sum_{j=1}^n \tilde{f}^-_j(\lambda)e_j := \frac{1}{\sqrt{2\pi}}\sum_{j=1}^n \big(f, U^-_{\lambda j}\big)_{\mathcal{H}} e_j$$

on elements $f = \langle \varphi^-, u, \varphi^+ \rangle$ in which $\varphi^-(\xi)$, $\varphi^+(\varsigma)$ and $u(x)$ are smooth compactly supported vector-valued functions. We give the lemma below.

Lemma 6. *The transformation $\mathcal{F}_-$ maps H_- isometrically onto $L_2((-\infty,\infty); E)$ and for all $f, g \in H_-$ the Parseval equality and the inversion formula are valid*:

$$(f,g)_{\mathcal{H}} = (\tilde{f}_-, \tilde{g}_-)_{L_2} = \int_0^\infty \sum_{j=1}^n \tilde{f}_j(\lambda)\overline{\tilde{g}_j(\lambda)}\,d\lambda,$$

$$f = \frac{1}{\sqrt{2\pi}} \int_0^\infty \sum_{j=1}^n U_{\lambda j}^- \tilde{f}_j^-(\lambda)\,d\lambda,$$

where $\tilde{f}_-(\lambda) = (\mathcal{F}_- f)(\lambda)$ and $\tilde{g}_-(\lambda) = (\mathcal{F}_- g)(\lambda)$.

Proof. For $f, g \in D_-$, $f = \langle \varphi^-, 0, 0 \rangle$ and $g = \langle \psi^-, 0, 0 \rangle$, with Paley–Wiener theorem, we have that

$$\tilde{f}_-(\lambda) = \frac{1}{\sqrt{2\pi}} \sum_{j=1}^n \bigl(f, U_{\lambda j}^-\bigr)_{\mathcal{H}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 \varphi^-(\xi) e^{i\lambda\xi}\,d\xi \in H_-^2(E),$$

and by using usual Parseval equality for Fourier integrals

$$(f,g)_{\mathcal{H}} = \int_{-\infty}^0 \bigl(\varphi^-(\xi), \psi^-(\xi)\bigr)_E\,d\xi = \int_{-\infty}^\infty \bigl(\tilde{f}_-(\lambda), \tilde{g}_-(\lambda)\bigr)_E\,d\lambda = (\mathcal{F}_- f, \mathcal{F}_- g)_{L_2}.$$

Here $H_\pm^2$ denote the Hardy classes in $L_2((-\infty,\infty); E)$ consisting of the vector-valued functions analytically extendible to the upper and lower half-planes, respectively. □

Now we can extend the Parseval equality to whole of H_-. Let us consider in H_- the dense set H'_- of elements obtained from smooth compactly supported vector-valued functions D_- as follows:

$$f \in H'_-: \quad f = \mathcal{U}_T f_0, \quad f_0 = \langle \varphi^-, 0, 0 \rangle, \quad \varphi^- \in C_0^\infty\bigl((-\infty, 0); E\bigr),$$

where $T = T_f$ depending on f is nonnegative number. If $f, g \in H_-$, then for $T > T_f$ and $T > T_g$ we have that $\mathcal{U}_{-T} f, \mathcal{U}_{-T} g \in D_-$ and the first components of these vectors belong to $C_0^\infty((-\infty,0); E)$. Thus, since the operators $\mathcal{U}_t$ are unitary, by using the equality

$$\mathcal{F}_- \mathcal{U}_t f = \frac{1}{\sqrt{2\pi}} \sum_{j=1}^n \bigl(\mathcal{U}_t f, U_{\lambda j}^-\bigr)_{\mathcal{H}} e_j = e^{i\lambda t} \mathcal{F}_- f,$$

we have

$$\begin{aligned}(f,g)_{\mathcal{H}} &= (\mathcal{U}_{-T} f, \mathcal{U}_{-T} g)_{\mathcal{H}} = (\mathcal{F}_- \mathcal{U}_{-T} f, \mathcal{F}_- \mathcal{U}_{-T} g)_{L_2} \\ &= (e^{-i\lambda t} \mathcal{F}_- f, e^{-i\lambda t} \mathcal{F}_- g)_{L_2} = (\mathcal{F}_- f, \mathcal{F}_- g)_{L_2}.\end{aligned} \tag{10}$$

Because the space H'_- denses in H_-, by taking closure in (10), the Parseval equality for the whole space H_- is obtained. If the all integrals in Parseval equality are considered as limits in the mean of integrals over fine intervals, the inversion formula is followed from Parseval equality. Finally,

$$\mathcal{F}_- H_- = \overline{\bigcup_{t \geqslant 0} \mathcal{F}_- \mathcal{U}_t D_-} = \overline{\bigcup_{t \geqslant 0} e^{-i\lambda t} H^2_-(E)} = L_2\big((-\infty, \infty); E\big),$$

that is $\mathcal{F}_-$ maps H_- onto the whole of $L_2((-\infty, \infty); E)$.

We set

$$U^+_{\lambda j}(x, \xi, \zeta) = \big\langle S_B e^{-i\lambda\xi} e_j, \theta(x, \lambda)\big(M_\infty(\lambda) + B^*\big)^{-1} C e_j, e^{-i\lambda\varsigma} e_j\big\rangle,$$
$$j = 1, 2, \ldots, n,$$

and define the transformation $\mathcal{F}_+ : f \to \tilde{f}_+(\lambda)$ by formula

$$(\mathcal{F}_+ f)(\lambda) := \tilde{f}_+(\lambda) := \sum_{j=1}^{n} \tilde{f}^+_j(\lambda) e_j := \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{n} \big(f, U^+_{\lambda j}\big)_{\mathcal{H}} e_j$$

on the elements $f = \langle \varphi^-, u, \varphi^+ \rangle$ in which $\varphi^-(\xi)$, $\varphi^+(\varsigma)$ and $u(x)$ are smooth compactly supported vector-valued functions.

Lemma 7. *The transformation $\mathcal{F}_+$ maps H_+ isometrically onto $L_2((-\infty, \infty); E)$, and for all $f, g \in H_+$ the Parseval equality and inversion formula are valid*:

$$(f, g)_{\mathcal{H}} = (\tilde{f}_+, \tilde{g}_+)_{L_2} = \int_0^\infty \sum_{j=1}^{n} \tilde{f}_j(\lambda) \overline{\tilde{g}_j(\lambda)}\, d\lambda,$$

$$f = \frac{1}{\sqrt{2\pi}} \int_0^\infty \sum_{j=1}^{n} U^+_{\lambda j} \tilde{f}^+_j(\lambda)\, d\lambda,$$

where $\tilde{f}_+(\lambda) = (\mathcal{F}_+ f)(\lambda)$ and $\tilde{g}_+(\lambda) = (\mathcal{F}_+ g)(\lambda)$.

Proof. The proof of this lemma is familiar with Lemma 6. □

Since $S_B(\lambda)$ is unitary for $-\infty < \lambda < \infty$, it follows from the explicit expression for the vectors $U^+_{\lambda j}$ and $U^-_{\lambda j}$ $(j = 1, 2, \ldots, n)$ that

$$U^+_{\lambda j} = \sum_{k=1}^{n} S_{jk}(\lambda) U^-_{\lambda k}, \quad j = 1, 2, \ldots, n, \tag{11}$$

where S_{jk} $(j = 1, 2, \ldots, n)$ are elements of the matrix $S_B(\lambda)$. According to Lemma 5, $\mathcal{H} = H_- = H_+$ from the above equality. Hence, property (3) of incoming and outgoing subspaces presented above has been established.

Thus, the transformation $\mathcal{F}_-$ maps H_- isometrically onto $L_2((-\infty, \infty); E)$; the subspace D_- is mapped onto $H^2_-(E)$, while the operator $\mathcal{U}_t$ go over into operators of multiplication by $e^{i\lambda t}$. This means that $\mathcal{F}_-$ is an incoming spectral representation of the group $\{\mathcal{U}_t\}$. Similarly, $\mathcal{F}_+$ is an outgoing spectral representation of $\{\mathcal{U}_t\}$. It follows from that the passage from the $\mathcal{F}_-$ representation of an element $f \in \mathcal{H}$ to its $\mathcal{F}_+$ representation is accomplished as $\tilde{f}_+(\lambda) = S_B^{-1}(\lambda)\tilde{f}_-(\lambda)$. According to [2], we have proved the following theorem.

Theorem 8. *The matrix $S_B^{-1}(\lambda)$ is the scattering matrix of the group $\{\mathcal{U}_t\}$ (of the operator $\mathcal{L}_B$).*

We set $K = \langle 0, H, 0\rangle$, so that $\mathcal{H} = D_- \oplus K \oplus D_+$. From the explicit form of the unitary transformation $\mathcal{F}_-$ it follows that under the mapping $\mathcal{F}_-$ we have

$$\begin{aligned}
&\mathcal{H} \to L_2\big((-\infty,\infty); E\big), \quad f \to \tilde{f}_-(\lambda) = (\mathcal{F}_- f)(\lambda),\\
&D_- \to H_-^2(E), \quad D_+ \to S_B H_+^2(E),\\
&K \to H_+^2(E) \ominus S_B H_+^2(E), \quad \mathcal{U}_t f \to \big(\mathcal{F}_- \mathcal{U}_t \mathcal{F}_-^{-1} \tilde{f}_-\big)(\lambda) = e^{i\lambda t}\tilde{f}_-(\lambda).
\end{aligned}$$

These formulas show that the operator $\tilde{L}_B(L_K)$ is unitary equivalent to the model dissipative operator with characteristic function $S_B(\lambda)$. We have thus proved the next theorem.

Theorem 9. *The characteristic function of the dissipative operator $\tilde{L}_B(L_K)$ coincides with the matrix-valued function $S_B(\lambda)$ defined in* (10).

4. Completeness theorem for system of eigenvectors and associated vectors of the dissipative operator

Problems of the spectral analysis of the dissipative operator $\tilde{L}_B(L_K)$ can be solved by using characteristic function. For example, the absence of a singular factor $S(\lambda)$ in the factorization $\det S_B(\lambda) = S(\lambda)B(\lambda)$ ensures the completeness of the system of eigenvectors and associated vectors of the operator $\tilde{L}_B(L_K)$ in the space $L_2((0,\infty); E)$ [2,8]. Here $B(\lambda)$ is the Blaschke product.

We first use the following lemma.

Lemma 10. *The characteristic function $\tilde{S}_K(\lambda)$ of the operator L_K has the form*

$$\begin{aligned}
\tilde{S}_K(\lambda) &:= S_B(\lambda)\\
&= X_1\big(I - K_1K_1^*\big)^{-1}\big(\theta(\xi) - K_1\big)\big(I - K_1^*\theta(\xi)\big)^{-1}\big(I - K_1^*K_1\big)^{1/2}X_2,
\end{aligned}$$

where $K_1 = -K$ is the Cayley transformation of the dissipative operator B and $\theta(\xi)$ is the Cayley transformation of the matrix-valued function $M_\infty(\lambda)$, $\xi = (\lambda - i)(\lambda + i)^{-1}$ and

$$\begin{aligned}
&X_1 := (\operatorname{Im} B)^{-1/2}(I - K_1)^{-1}\big(I - K_1K_1^*\big)^{1/2},\\
&X_2 := \big(I - K_1^*K_1\big)^{-1/2}\big(I - K_1^*\big)(\operatorname{Im} B)^{1/2},\\
&|\det X_1||\det X_2| = 1.
\end{aligned}$$

It is known [2,9] that the inner matrix-valued function $\tilde{S}_K(\lambda)$ is a Blaschke–Potapov product if and only if $\det \tilde{S}_K(\lambda)$ is a Blaschke product. Then, from Lemma 10 it follows that the characteristic function $\tilde{S}_K(\lambda)$ is a Blaschke–Potapov product if and only if the matrix-valued function

$$X_K(\xi) = \left(I - K_1K_1^*\right)^{-1/2}\left(\theta(\xi) - K_1\right)\left(I - K_1^*\theta(\xi)\right)^{-1}\left(I - K_1^*K_1\right)^{1/2}$$

is a Blaschke–Potapov product in the unit disk.

In order to state the completeness theorem, we first define a suitable form for the Γ-capacity [9,10].

Let $\tilde{E}$ be an m-dimensional ($m < \infty$) Hilbert space. In $\tilde{E}$ we fix an orthonormal basis $e_1, e_2, \dots, e_m$ and denote by E_k ($k = 1, 2, \dots, m$) the linear span of vectors $e_1, e_2, \dots, e_k$. If $M \subset E_k$, then the population of $x \in E_{k-1}$ with the property $\mathrm{Cap}\{\lambda\colon \lambda \in \mathbb{C}, (x + \lambda e_k) \subset M\} > 0$ will be shown by $\Gamma_{k-1}M$ ($\mathrm{Cap}\, G$ is the inner logarithmic capacity of a set $G \subset \mathbb{C}$). The Γ-capacity of a set $M \subset \tilde{E}$ is a number

$$\Gamma - \mathrm{Cap}\, M := \sup \mathrm{Cap}\{\lambda\colon\ \lambda \in \mathbb{C},\ \lambda e_1 \subset \Gamma_1\Gamma_2 \dots \Gamma_{m-1}M\},$$

where supremum is taken with respect to all orthonormal basics in $\tilde{E}$. It is known [10] that every set $M \subset \tilde{E}$ of zero Γ-capacity has zero $2m^2$-dimensional Lebesgue measure, however the converse is not true [3].

Denote by $[E]$ the set of all linear operators in E. To convert $[E]$ into an n^2-dimensional Hilbert space, we give the inner product $\langle T, S\rangle = \mathrm{tr}\, S^*T$ for $T, S \in [E]$ ($\mathrm{tr}\, S^*T$, is the trace of the operator S^*T). Hence we may give the Γ-capacity of a set in $[E]$.

We use the following important result of [9].

Lemma 11. *Let $X(\xi)$ ($|\xi| < 1$) be a holomorphic function with the values to be contractive operators in $[E]$ ($\|X(\xi)\| \leqslant 1$). Then for Γ-quasi-every strictly contractive operators (i.e., for all strictly contractive $K \in [E]$ possible with the exception of a set of Γ of zero capacity) the inner part of the contractive function*

$$X_K(\xi) := (I - KK^*)^{-1/2}\left(X(\xi) - K\right)\left(I - K^*X(\xi)\right)^{-1}(I - K^*K)^{1/2}$$

is a Blaschke–Potapov product.

By summing all obtained result for the dissipative operator $L_K(\tilde{L}_B)$ we have proved following theorem.

Theorem 12. *For Γ-quasi-every strictly contractive $K \in [E]$ the characteristic function $\tilde{S}_K(\lambda)$ of the dissipative operator L_K is a Blaschke–Potapov product and spectrum of L_K is purely discrete and belongs to the open upper half-plane. For Γ-quasi-every strictly contractive $K \in [E]$ the operator L_K has an countable number of isolated eigenvalues with finite multiplicity and limit points at infinity, and the system of eigenvectors and associated vectors of this operators is complete in $L_2((-\infty, \infty); E)$.*

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